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Transformations of the Compressible Turbulent Boundary Layer with Heat Exchange

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A mathematical transformation is presented which converts the integral turbulent boundary-layer equations at high speed into one at low speed. The transformation is inspired by that recently introduced by Coles for the constant-pressure case. Here it is developed for the general case when pressure gradients, as well as heat exchange, are present. It is shown that in this case two transformations can be defined: an *i* transformation, in which the low-speed fluid is incompressible, and an *l* transformation, in which the low-speed fluid is a gas. The conditions under which the transformations operate correctly, not only on the integral equations but also on the whole system of partial differential equations, are discussed.

Nomenclature

x = tangential coordinate
y,y' = normal coordinate
ρ,p,T = density, pressure, and temperature
h,h^0 = static and stagnation enthalpy
μ,λ = viscosity and heat conduction coefficients
Pr = Prandtl number
u,u' = tangential velocity component
v = normal velocity component
m = mass flux per unit width
I = momentum flux, per unit width
H,H^0 = enthalpy and stagnation enthalpy flux per unit width
ψ,ψ' = stream function per unit width or per radian
δ = boundary-layer thickness
δ^* = displacement thickness
θ = momentum thickness
φ = energy thickness
φ^* = density thickness
τ = viscous stress per unit width
q = heat flux per unit width
r = distance of wall from axis (axisymmetric flows); or recovery factor (3.6)
ϵ = exponent = 0 (two-dimensional flow) or = 1 (axisymmetric flow)
c_f = skin-friction coefficient
c_h = Stanton number
s = Reynolds analogy factor (3.6)
$\sigma(x),\eta(x),\zeta(x)$ = functions defining the transformation
$c(x),d(x)$ = coefficients of the enthalpy-velocity expansion
Re_ψ,Re_s = Reynolds numbers
$\bar{u}_\tau$ = incompressible friction velocity

Subscripts

e = external flow
w = wall
r = recovery value
m = mean
s = substructure

Superscripts

0 = stagnation
— = low-speed flow

1. Introduction

Any treatment of the turbulent boundary layer contains an element of arbitrariness. Our very incomplete knowledge of the turbulent mechanism in shear motion prevents the development of a theory based on firm rational grounds, and the only courses presently available are the result of either some shaky physical concept or some not completely justifiable mathematical formalism. This is true for the most elementary type of turbulent boundary-layer flow, for which density, pressure, and viscosity are constant, as well as for the more complicated cases where the characteristic parameters of the fluid are variable as a result of pressure gradients, heat exchange, and high flow velocities. However, while in the former case the physical concepts and the mathematical formalism involved have received the blessing of empirical confirmation from a very large number of experimental results, this is not true in the latter case. As soon as the compressibility effects start creeping in, ambiguities appear in the way of extending the same physical concepts that were empirically justified for the most elementary case. Similar ambiguities present themselves in the more mathematical approach, and there does not appear to be a completely satisfactory way out from these ambiguities.

2. Mathematical Transformations

The result of this uncomfortable situation is that the choice of a particular path is a matter of personal preference (call it intuition).† Like many other authors, this author feels particularly attracted by the mathematical approach, of

† Examples of the first approach (extension of the empirically proved physical concepts) are in Refs. 1–5. Examples of the second approach (mathematical transformation) are in Refs. 6–10. These references are far from being complete; they have only the purpose of illustration, or they are referred to in what follows.

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which the ultimate goal is to establish a correspondence between a given flow containing one or more disturbing elements (such as nonvanishing pressure gradients, heat exchange, or high Mach numbers) and another, simpler physical flow where some of the disturbing elements are absent. This can be accomplished by using appropriate mathematical transformations which are able to suppress one or more of the disturbing elements. If the behavior of the simplified flow is known, the application to it of the reciprocal transformation will result in the prediction of a certain behavior for the original flow. At this point, the assumption can be made that this is the actual physical behavior of the flow. Up to a certain point, this procedure is the same as that expressed by the Stewartson-Illingworth transformation for the correspondence between a high Mach number laminar boundary-layer flow and a low-speed flow. The Stewartson-Illingworth transformation produces an exact mathematical correspondence between two adiabatic flows or between two flows with heat exchange involving a perfect gas of Prandtl number unity and viscosity proportional to the absolute temperature, and there is no need of a particular assumption to establish the validity of this correspondence. The difference in the turbulent case comes from the following fact: since the mechanism determining the distribution of the turbulent shearing stress is unknown, one cannot be assured of the physical validity of the transformation, even if this correctly transforms the inertia and pressure terms of the equations. Coles[10] has recently tried to ennoble this mathematical procedure by inverting the statement and saying that, if a mathematical transformation for the inertia and pressure terms can be found and the transformation is "physically realistic," then, also, the shearing stresses are correctly treated by the transformation. The way this author interprets Coles' statement is that, if the factors of the transformation can be adjusted so as to provide a satisfactory correspondence between the behavior of a properly selected individual quantity in the two flows (such as the friction coefficient), it follows then that the transformation operates correctly not only at the wall, where the friction coefficient is determined, but throughout the boundary layer. This statement can hardly be accepted as a proof, and actually this author has reasonable doubts that it is correct. Fortunately, the results of Coles depend only to a minor extent on the exact validity of his statement‡; actually, most of them can be obtained without making use of it and by following the more conventional line of looking for a transformation of the integral equations for the whole boundary layer rather than that of the local partial differential equations.

All transformations that have been used are genetically related to the Howarth-Dorodnitsin transformation which replaces the transverse coordinate y with one proportional to

$$\int_0^y \rho dy$$

The Stewartson-Illingworth transformation introduces not only an additional transformation of the longitudinal coordinate x, but also a factor which is a function of x, in the y transformation. Several authors have elaborated on this transformation to adapt it to turbulent flows. One element common to allt hese transformations is that, although the coordinates and the velocity components are transformed, the stream function is invariant with respect to the transformation. It has been the merit of Coles to remove this restriction, probably unpremeditated rather than deliberate, hence giving to the transformation one additional degree of freedom. The procedure that follows is to a large extent inspired by that of Coles, although it runs along a slightly different path and reaches probably somewhat deeper.

‡ In fact, only the developments at the end of Sec. 5 depend to a certain extent on this statement.

Instead of starting, as Coles did, from the differential equations of the flow this author starts from the integral momentum and energy equations. The mass, momentum, and stagnation enthalpy flux per unit width can be written in the original physical plane:

$$m = \int_0^\delta \rho u dy \qquad I = \int_0^\delta \rho u^2 dy \qquad H^0 = \int_0^\delta \rho u h^0 dy \tag{2.1}$$

(Here δ can be chosen to be the larger of the velocity and the thermal thicknesses, when these two are not equal.) One can, as usual, define the displacement, the momentum, and the stagnation enthalpy thicknesses through the equations

$$\rho_e u_e \delta - m = \rho_e u_e \delta^* \qquad m u_e - I = \rho_e u_e^2 \theta$$
$$m h_e^0 - H^0 = \rho_e u_e (h_e^0 - h_w)\varphi \tag{2.2}$$

so that the three thicknesses are explicitly obtained as

$$\delta^* = \int_0^\delta \left(1 - \frac{\rho u}{\rho_e u_e}\right) dy \qquad \theta = \int_0^\delta \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u}{u_e}\right) dy$$
$$\varphi = \int_0^\delta \frac{\rho u}{\rho_e u_e} \frac{h_e^0 - h^0}{h_e^0 - h_w} dy \tag{2.3}$$

Here the integration could be extended to ∞ rather than δ without affecting the value of the integrals.

The conservation of momentum and total energy for the boundary layer as a whole can be written §

$$\frac{d}{dx}(r^\epsilon I) = u_e \frac{d}{dx}(r^\epsilon m) - r^\epsilon \delta \frac{dp}{dx} - r^\epsilon \tau_w$$
$$\frac{d}{dx}(r^\epsilon H^0) = h_e^0 \frac{d}{dx}(r^\epsilon m) - r^\epsilon q_w \tag{2.4}$$

The factor r^ϵ has been introduced, as usual, to include in a single formulation both the case of plane flow ($\epsilon = 0$) and that of axisymmetric flow ($\epsilon = 1$) (in which case the equations provide the balance per radian). In the latter case, r represents the distance of the wall from the axis of symmetry, and the assumption $\delta \ll r$ is implicit.

Making use of the definitions (2.2), Eqs. (2.4) can be rewritten as

$$\frac{d}{dx}(r^\epsilon \rho_e u_e^2 \theta) - r^\epsilon \delta^* \frac{dp}{dx} = r^\epsilon \tau_w \tag{2.5}$$

$$\frac{d}{dx}[r^\epsilon \rho_e u_e (h_e^0 - h_w)\varphi] = r^\epsilon q_w \tag{2.6}$$

Here use has been made of the fact that $h_e^0 = \text{const}$ and of the Bernoulli relation in the external flow:

$$dp/dx = -\rho_e u_e (du_e/dx) \tag{2.7}$$

Let us consider now a low-speed flow, of an as yet unspecified fluid. Following Coles, we shall indicate with a bar the corresponding quantities, and write the equations corresponding to Eqs. (2.1–2.3, 2.5, and 2.6) as follows:

$$\bar{m} = \int_0^{\bar\delta} \bar\rho \bar u d\bar y; \quad I = \int_0^{\bar\delta} \bar\rho \bar u^2 d\bar y; \quad \bar H = \int_0^{\bar\delta} \bar\rho \bar u \bar h d\bar y \tag{2.1a}$$

$$\bar\rho_e \bar u_e \bar\delta - \bar m = \bar\rho_e \bar u_e \bar\delta^*$$
$$\bar m \bar u_e - I = \bar\rho_e \bar u_e^2 \bar\theta, \bar m \bar h_e - \bar H = \bar\rho_e \bar u_e (\bar h_e - \bar h_w)\bar\varphi \tag{2.2a}$$

§ Clearly it is assumed in writing the energy equation that no other form of energy is exchanged. At hypersonic velocities, doubts may arise about the validity of neglecting the increasing level of acoustic energy radiated from high-speed turbulent flows. No evaluation of this level has been attempted, to my knowledge, in relation to boundary-layer flows.

$$\bar{\delta}^* = \int_0^{\bar{\delta}} \left(1 - \frac{\bar{\rho}\bar{u}}{\bar{\rho}_e\bar{u}_e}\right) d\bar{y} \qquad \bar{\theta} = \int_0^{\bar{\delta}} \frac{\bar{\rho}\bar{u}}{\bar{\rho}_e\bar{u}_e}\left(1 - \frac{\bar{u}}{\bar{u}_e}\right) d\bar{y}$$

$$\bar{\varphi} = \int_0^{\bar{\delta}} \frac{\bar{\rho}\bar{u}}{\bar{\rho}_e\bar{u}_e}\frac{\bar{h}_e - \bar{h}}{\bar{h}_e - \bar{h}_w} d\bar{y} \tag{2.3a}$$

$$\frac{d}{d\bar{x}}(\bar{r}^\epsilon\bar{\rho}_e\bar{u}_e{}^2\bar{\theta}) - \bar{r}^\epsilon\bar{\delta}^*\frac{d\bar{p}}{d\bar{x}} = \bar{r}^\epsilon\bar{\tau}_w \tag{2.5a}$$

$$\frac{d}{d\bar{x}}[\bar{r}^\epsilon\bar{\rho}_e\bar{u}_e(\bar{h}_e - \bar{h}_w)\bar{\varphi}] = \bar{r}^\epsilon\bar{q}_w \tag{2.6a}$$

$$\frac{d\bar{p}}{d\bar{x}} = -\bar{\rho}_e\bar{u}_e\frac{d\bar{u}_e}{d\bar{x}} \tag{2.7a}$$

Observe that in view of the low velocity the stagnation enthalpy has been replaced by the static enthalpy.

Now look for a transformation which, when applied to Eqs. (2.5) and (2.6), will result in (2.5a) and (2.6a). For this purpose we introduce the stream functions,

$$\psi = r^\epsilon\int_0^y \rho u dy' \qquad \bar{\psi} = \bar{r}^\epsilon\int_0^{\bar{y}} \bar{\rho}\bar{u}\, d\bar{y}' \tag{2.8}$$

observing that at the edge of the boundary layer ψ and $\bar{\psi}$ coincide with mr^ϵ and $\bar{m}\bar{r}^\epsilon$. With Coles we introduce the transformation

$$\left(\frac{r}{\bar{r}}\right)^\epsilon\frac{\bar{\psi}(\bar{x},\bar{y})}{\psi(x,y)} = \sigma(x) \qquad \frac{\bar{\rho}}{\rho}\frac{\partial\bar{y}}{\partial y} = \eta(x) \qquad \frac{d\bar{x}}{dx} = \xi(x) \tag{2.9}$$

From (2.8) and the first two (2.9) one obtains the following correspondence between the velocities:

$$\bar{u}/u = \sigma/\eta = \bar{u}_e/u_e \tag{2.10}$$

Substitution of the second (2.9) and (2.10) into (2.3a) and comparison with (2.3) shows that

$$\bar{\theta} = \frac{\eta\rho_e}{\bar{\rho}_e}\theta \qquad \varphi = \frac{\eta\rho_e}{\bar{\rho}_e}\bar{\varphi} \tag{2.11}$$

the second relation being true provided we associate to the transformation (2.9) the following correspondence:

$$\bar{h}/\bar{h}_e = h^0/h_e{}^0 \tag{2.12}$$

The relation between displacement thicknesses is also obtained from the same equations in the form

$$\bar{\delta}^* = \frac{\eta\rho_e}{\bar{\rho}_e}\left[\delta^* - \int_0^\delta\left(1 - \frac{\bar{\rho}_e}{\bar{\rho}}\frac{\rho}{\rho_e}\right)dy\right] \tag{2.13}$$

Introducing (2.10) and the last (2.9) in (2.7a) and comparing with (2.7), one obtains

$$\frac{d\bar{p}}{d\bar{x}} = \frac{\sigma^2}{\xi\eta^2}\frac{\bar{\rho}_e}{\rho_e}\left[\frac{dp}{dx} + \rho_e u_e{}^2\frac{d}{dx}\left(\ln\frac{\eta}{\sigma}\right)\right] \tag{2.14}$$

Now Eqs. (2.9–2.11, 2.13, and 2.14) can be introduced in (2.5a). Comparison with (2.5) results in the following correspondence between skin frictions. (This relation and others to follow are the same for the two-dimensional and the axisymmetric case.)

$$\bar{\tau}_w = \frac{\sigma^2}{\xi\eta}\left\{\tau_w + \rho_e u_e{}^2\theta\frac{d}{dx}(\ln\sigma) - \rho_e u_e{}^2\left[\delta^* + \theta - \int_0^\delta\left(1 - \frac{\bar{\rho}_e}{\bar{\rho}}\frac{\rho}{\rho_e}\right)dy\right]\frac{d}{dx}\left(\ln\frac{\eta}{\sigma}\right) + \frac{dp}{dx}\int_0^\delta\left(1 - \frac{\bar{\rho}_e}{\bar{\rho}}\frac{\rho}{\rho_e}\right)dy\right\} \tag{2.15}$$

provided the following geometric condition is satisfied:

$$\left[\frac{\bar{r}(\bar{x})}{r(x)}\right]^\epsilon = \text{const} \tag{2.15a}$$

(2.15a) is an effective condition only in the axisymmetric case, $\epsilon = 1$.

On the other hand, assuming Newtonian friction at the wall, that is,

$$\tau_w = \mu_w\left(\frac{\partial u}{\partial y}\right)_w \qquad \bar{\tau}_w = \bar{\mu}_w\left(\frac{\partial\bar{u}}{\partial\bar{y}}\right)_w \tag{2.16}$$

and making use of (2.9) and (2.10), one gets

$$\frac{\bar{\tau}_w}{\tau_w} = \frac{\sigma}{\eta^2}\frac{\bar{\rho}_w\bar{\mu}_w}{\rho_w\mu_w} \tag{2.17}$$

Hence, if one rewrites (2.15) in the form

$$\xi = \frac{\sigma^2}{\eta}\frac{\tau_w}{\bar{\tau}_w}\left[1 + \frac{2}{c_f}\left\{\theta\frac{d}{dx}(\ln\sigma) - \left[\delta^* + \theta - \int_0^\delta\left(1 - \frac{\bar{\rho}_e}{\bar{\rho}}\frac{\rho}{\rho_e}\right)dy\right]\frac{d}{dx}\left(\ln\frac{\eta}{\sigma}\right) - \frac{1}{u_e}\frac{du_e}{dx}\int_0^\delta\left(1 - \frac{\bar{\rho}_e}{\bar{\rho}}\frac{\rho}{\rho_e}\right)dy\right\}\right] \tag{2.18}$$

we see that if $\sigma(x)$ and $\eta(x)$ are known, $\xi(x)$ is also determined by (2.18) and (2.17). Here the friction coefficient c_f is defined, as usual, by

$$\tau_w = \tfrac{1}{2}c_f\rho_e u_e{}^2 \tag{2.19}$$

A similar treatment can be applied to Eqs. (2.6a) and (2.6). The result, for h_w = const and subject to the condition (2.15a), is

$$\bar{q}_w = \frac{\sigma}{\xi}\frac{\bar{h}_e}{h_e{}^0}\left[q_w + \rho_e u_e(h_e{}^0 - h_w)\varphi\frac{d}{dx}(\ln\sigma)\right] \tag{2.20}$$

At the wall, where

$$q_w = \lambda_w\left(\frac{\partial T}{\partial y}\right)_w = \frac{\mu_w}{Pr_w}\left(\frac{\partial h}{\partial y}\right)_w = \frac{\mu_w}{Pr_w}\left(\frac{\partial h^0}{\partial y}\right)_w$$

$$\bar{q}_w = \bar{\lambda}_w\left(\frac{\partial\bar{T}}{\partial\bar{y}}\right)_w = \frac{\bar{\mu}_w}{\overline{Pr}_w}\left(\frac{\partial\bar{h}}{\partial\bar{y}}\right)_w \tag{2.21}$$

we obtain, with the help of (2.9) and (2.12) and assuming $Pr_w = \overline{Pr}_w$,

$$\frac{\bar{q}_w}{q_w} = \frac{1}{\eta}\frac{\bar{h}_e}{h_e{}^0}\frac{\bar{\rho}_w\bar{\mu}_w}{\rho_w\mu_w} \tag{2.22}$$

Hence, if one rewrites (2.20) in the form

$$\xi = \sigma\frac{\bar{h}_e}{h_e{}^0}\frac{q_w}{\bar{q}_w}\left[1 + \frac{h_e{}^0 - h_w}{h_r - h_w}\frac{\varphi}{c_h}\frac{d}{dx}(\ln\sigma)\right] \tag{2.23}$$

where c_h is the Stanton number defined by

$$q_w = \rho_e u_e(h_r - h_w)c_h \tag{2.24}$$

one has a second relation defining $\xi(x)$ once $\sigma(x)$ is known. Clearly, the consistency of the transformation requires the two expressions (2.18) and (2.23) for $\xi(x)$ to coincide, and since the factor in front of the brackets is identical in both expressions, by virtue of (2.17) and (2.22), the following expression is obtained by equating the expressions within the brackets:

$$\left[\delta^* + \theta - \int_0^\delta\left(1 - \frac{\bar{\rho}_e}{\bar{\rho}}\frac{\rho}{\rho_e}\right)dy\right]\frac{d}{dx}\left(\ln\frac{\eta}{\sigma}\right) = \left[\theta - \frac{h_e{}^0 - h_w}{h_r - h_w}\frac{c_f}{2c_h}\varphi\right]\frac{d}{dx}(\ln\sigma) - \frac{1}{u_e}\frac{du_e}{dx}\int_0^\delta\left(1 - \frac{\bar{\rho}_e}{\bar{\rho}}\frac{\rho}{\rho_e}\right)dy \tag{2.25}$$

For a given $\sigma(x)$, $\eta(x)$ can be obtained upon integration of this expression for any particular behavior of the assigned boundary-layer flow. Then the value of $\xi(x)$ can be calculated from either (2.18) or (2.23) with the help of (2.17) or (2.22).

In order for the transformation to be entirely defined, one still needs the knowledge of $\sigma(x)$ and of the state properties of the fluid flowing in the low-speed flow. These problems will be discussed in the following sections.‖ We observe that the integration of (2.25) provides η/σ to within an arbitrary constant factor. According to (2.10) this means that the velocity scale between the two flows can be chosen arbitrarily at one point of the flow, after which it is entirely determined at any other point. Once $\sigma(x)$, $\eta(x)$, $\xi(x)$ are determined and the properties of the low-velocity fluid are chosen, the transformation (2.9) is completely determined. The transformation can now be applied to the differential equations of motion, as Coles did. The momentum equations of the two flows are

$$\left.\begin{aligned} \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} &= -\frac{dp}{dx} + \frac{\partial \tau}{\partial y} \\ \bar\rho \bar u \frac{\partial \bar u}{\partial \bar x} + \bar\rho \bar v \frac{\partial \bar u}{\partial \bar y} &= -\frac{d\bar p}{d\bar x} + \frac{\partial \bar\tau}{\partial \bar y} \end{aligned}\right\} \quad (2.26)$$

The value of $\partial\bar\tau/\partial\bar y$, as obtained from the second (2.26), is transformed by (2.9), making use of (2.14) also. Upon integration of the resulting expression with respect to y, one obtains

$$\bar\tau = \frac{\sigma^2}{\xi\eta}\left[\tau + \frac{1}{r^\epsilon}\frac{d}{dx}(\ln\sigma)\int_y^\delta \psi\frac{\partial u}{\partial y}\,dy + \rho_e u_e^2\frac{d}{dx}\left(\ln\frac{\eta}{\sigma}\right)\int_y^\delta\frac{\rho}{\rho_e}\left(\frac{u^2}{u_e^2} - \frac{\bar\rho_e}{\bar\rho}\right)dy + \frac{dp}{dx}\int_y^\delta\left(1 - \frac{\bar\rho_e}{\bar\rho}\frac{\rho}{\rho_e}\right)dy\right] \quad (2.27)$$

The integration constant has been chosen so as to produce vanishing $\bar\tau$ and τ at the edge of the boundary layer.

The energy equations for the two flows are [see footnote to Eq. (2.4)]

$$\rho u\frac{\partial h^0}{\partial x} + \rho v\frac{\partial h^0}{\partial y} = \frac{\partial}{\partial y}(q + u\tau);\ \bar\rho\bar u\frac{\partial\bar h}{\partial\bar x} + \bar\rho\bar v\frac{\partial\bar h}{\partial\bar y} = \frac{\partial\bar q}{\partial\bar y} \quad (2.28)$$

Proceeding for the $\bar q$ as previously with the $\bar\tau$, one obtains, taking into account (2.12),

$$\bar q = \frac{\sigma}{\xi}\frac{\bar h_e}{h_e^0}\left[q + u\tau + \frac{1}{r^\epsilon}\frac{d}{dx}(\ln\sigma)\int_y^\delta\psi\frac{\partial h^0}{\partial y}\,dy\right] \quad (2.29)$$

For $y = 0$, the relation (2.27) reduces to (2.15), and (2.29) to (2.20); hence nothing new is obtained concerning the transformation of the integral equations. The new result implied in (2.27) and (2.29) is that, if the values they produce for $\bar\tau$ and $\bar q$ are consistent with the physical behavior of the low-velocity boundary-layer flow, then the transformation has the higher power of establishing a correspondence not only in the integral behavior of the two flows, but also in their local behavior.

‖ We shall see in Sec. 4 that there is a certain arbitrariness in the choice of the low-speed fluid. On the other hand, the choice of $\sigma(x)$ cannot be made without the help and interpretation of empirical evidence. This was done by Coles for the constant pressure case, resulting in his introduction of the concept of turbulent substructure. Though the physical interpretation of this concept is nebulous, its application produces an extremely satisfactory correlation of the friction coefficients in the constant-pressure adiabatic case. If desirable experimental checks on high-speed or low-speed flow with heat exchange will also result in an accurate correlation, Cole's concept can, in my mind, safely be extended to the variable-pressure case. In Sec. 5 I have presented my personal derivation of $\sigma(x)$ from Coles' concept of a substructure.

3. Constant-Pressure Case

The only case treated by Coles is the constant-pressure case, for which ρ_e and u_e are constant. The constant-pressure case is consistent only with $\epsilon = 0$ or r = const; hence from Eqs. (2.5) and (2.6), taking Eqs. (2.19) and (2.24) into account, we derive the following relation:

$$\frac{d\varphi}{d\theta} = \frac{h_r - h_w}{h_e^0 - h_w}\frac{2c_h}{c_f} \quad (3.1)$$

It is clear that, if the second member of this relation is constant, and the thermal and velocity layer start building up at the same station, both terms on the right-hand side of (2.25) vanish and this can be integrated to η/σ = const. This is indeed the relation obtained by Coles from the consideration of Eq. (2.14) and the postulate that the two corresponding flows must both present vanishing pressure gradients.

This result is certainly correct when $Pr_t = 1$, since then $h_r = h_e^0$ and $c_h = c_f/2$ (as can be found immediately from the fact that the stagnation enthalpy is a linear function of the velocity[11]), so that the second member of (3.1) is unity and $\varphi = \theta$ (if both vanish at the same station).

The subscript t added to the Prandtl number is to indicate that, if the flow is turbulent, the relation between heat exchange and frictional stresses is not related to the "molecular" Prandtl number, except near the wall. The definition of the "turbulent" Prandtl number is then obtained from the generalized Reynolds relation

$$\frac{q}{\tau} = \frac{1}{Pr_t}\frac{(\partial h/\partial y)}{(\partial u/\partial y)} \quad (3.2)$$

At and near the wall, $Pr_t \to Pr$. To discuss the case $Pr_t \neq 1$, it is appropriate to write the differential energy equation in the form

$$\tau\frac{\partial^2 h}{\partial u^2} + (1 - Pr_t)\frac{\partial\tau}{\partial u}\frac{\partial h}{\partial u} + Pr_t\frac{dp}{dx}\frac{\partial y}{\partial u}\left(u + \frac{\partial h}{\partial u}\right) - Pr_t\,\rho u\frac{\partial y}{\partial u}\frac{\partial h}{\partial x} = 0 \quad (3.3)$$

which can be obtained[12] by switching to u and x as independent variables, instead of y and x. For the constant-pressure case, and if h is a function of u alone, this equation simplifies to

$$\tau\frac{d^2h}{du^2} + (1 - Pr_t)\frac{\partial\tau}{\partial u}\frac{dh}{du} = 0$$

and can be integrated, for the ideal case Pr_t = const, to produce

$$h = h_w + u_e\left(\frac{dh}{du}\right)_w F_1\left(\frac{u}{u_e}\right) - u_e^2 F_2\left(\frac{u}{u_e}\right) \quad (3.4)$$

with

$$F_1(z) = \int_0^z\left(\frac{\tau}{\tau_w}\right)^{Pr_t - 1} d\left(\frac{u}{u_e}\right)$$

$$F_2(z) = Pr_t\int_0^z\left(\frac{\tau}{\tau_w}\right)^{Pr_t - 1} d\left(\frac{u}{u_e}\right)\int_0^{u/u_e}\left(\frac{\tau}{\tau_w}\right)^{1 - Pr_t} d\left(\frac{u'}{u_e}\right) \quad (3.5)$$

Clearly the assumption that h is a function of u alone is fulfilled only if τ/τ_w is one also. The relation (3.4) was developed originally for laminar boundary layers, for which this condition is satisfied. Van Driest[13] applied the same relation to the turbulent boundary-layer case, for which the condition is not exactly satisfied. However, it must be noticed that, if Pr_t is sufficiently close to unity, since τ/τ_w must anyway go from 1 to 0 as u/u_e goes from 0 to 1, the functions (3.5) are rather insensitive to small variations of the τ distribution, and the assumption that h is a function of u alone must be almost exactly, though not exactly, fulfilled.

If (3.3) is accepted, simple elaborations allow the derivations of the recovery factor r and of the Reynolds analogy factor s in the form

$$r = \frac{h_r - h_e}{h_e^0 - h_e} = 2F_2(1); \qquad s = Pr_t \frac{h_r - h_w}{u_e(dh/du)_w} = Pr_t F_1(1) \tag{.6}$$

From (3.6) one can calculate

$$\frac{c_f}{2c_h} = s$$
$$\frac{h_r - h_w}{h_e^0 - h_w} = r + (1-r)\frac{h_e - h_w}{h_e^0 - h_w} \tag{3.7}$$

Hence, if the condition that h is a function of u alone is almost exactly fulfilled, since both expressions (3.7) are constant, so is the second member of (3.1), and Coles' result η/σ = const is also almost exactly true. This result can be safely extended to the more realistic case when Pr_t, though close to unity, is not constant. Hence it is also almost exactly true that the condition of constant pressure is invariant against the transformation, as originally postulated by Coles.

Next, it is interesting to check if the transformation that we have derived, considering the integral boundary-layer equations, also holds locally for the equations in differential form, as required by Coles. This means that the values of $\bar{\tau}$ and $\bar{q}$ as obtained from (2.27) and (2.29), that is,

$$\bar{\tau} = \frac{\sigma^2}{\xi\eta}\left[\tau + \frac{1}{r^\epsilon}\frac{d}{dx}(\ln\sigma)\int_y^\delta \psi \frac{\partial u}{\partial y}dy\right]$$
$$\bar{q} = \frac{\sigma}{\xi}\frac{\bar{h}_e}{h_e^0}\left[q + u\tau + \frac{1}{r^\epsilon}\frac{d}{dx}(\ln\sigma)\int_y^\delta \psi\frac{\partial h^0}{\partial y}dy\right] \tag{3.8}$$

must be consistent with the physical reality of the transformed flow. In addition, the transverse velocity in the transformed flow must satisfy the relation

$$\bar{v} = \frac{\sigma}{\xi}\left[\frac{\rho}{\bar{\rho}}v + \frac{u}{\eta}\frac{\partial}{\partial x}\left(\eta\int_0^y \frac{\rho}{\bar{\rho}}dy\right) - \frac{\psi}{\bar{\rho}r^\epsilon}\frac{d}{dx}(\ln\sigma)\right] \tag{3.9}$$

obtained from the definition equations

$$r^\epsilon \rho v = -\frac{\partial\psi}{\partial x} \qquad \bar{r}^\epsilon\bar{\rho}\bar{v} = -\frac{\partial\bar{\psi}}{\partial\bar{x}}$$

after using the transformation (2.9).

There is certainly no difficulty about satisfying (3.9). About the consistency of the relations (3.8), one observes that the physical reality of the transformed flow requires that the Reynolds analogy (3.2) applies to this flow as well as to the original flow; hence,

$$\frac{q}{\tau} = \frac{1}{Pr_t}\frac{(\partial h/\partial y)}{(\partial u/\partial y)} \qquad \frac{\bar{q}}{\bar{\tau}} = \frac{1}{\overline{Pr}_t}\frac{(\partial\bar{h}/\partial\bar{y})}{(\partial\bar{u}/\partial\bar{y})} \tag{3.10}$$

The second (3.10) can be transformed with the help of (2.10) and (2.12) and compared with the same quantity as obtained from (3.8), thus providing

$$\frac{\bar{q}}{\bar{\tau}} = \frac{1}{\overline{Pr}_t}\frac{\eta}{\sigma}\frac{\bar{h}_e}{h_e^0}\frac{\partial h^0/\partial y}{\partial u/\partial y}$$
$$= \frac{\eta}{\sigma}\frac{\bar{h}_e}{h_e^0}\frac{q + u\tau + \dfrac{1}{r^\epsilon}\dfrac{d}{dx}(\ln\sigma)\displaystyle\int_y^\delta\psi\frac{\partial h^0}{\partial y}dy}{\tau + \dfrac{1}{r^\epsilon}\dfrac{d}{dx}(\ln\sigma)\displaystyle\int_y^\delta\psi\frac{\partial u}{\partial y}dy} \tag{3.11}$$

On the other hand, the first (3.10) gives

$$q = \frac{\tau}{Pr_t}\left(\frac{(\partial h^0/\partial y)}{(\partial u/\partial y)} - u\right) \tag{3.12}$$

Finally, from (3.11) and (3.12) one obtains

$$\frac{1}{\overline{Pr}_t}\left(\tau + \frac{1}{r^\epsilon}\frac{d}{dx}(\ln\sigma)\int_y^\delta\psi\frac{\partial u}{\partial y}dy\right) = \left[\frac{1}{Pr_t} - \left(\frac{1}{Pr_t} - 1\right)u\frac{(\partial u/\partial y)}{(\partial h^0/\partial y)}\right]\tau \times \frac{1}{r^\epsilon}\frac{d}{dx}(\ln\sigma)\frac{(\partial u/\partial y)}{(\partial h^0/\partial y)}\int_y^\delta\psi\frac{\partial h^0}{\partial y}dy \tag{3.13}$$

It is immediately clear that, if $Pr_t = 1$, Eq. (3.13) is identically satisfied by $\overline{Pr}_t = 1$, since then $(\partial u/\partial y)/(\partial h^0/\partial y)$ is a constant. If, however, Pr_t is different from unity, and if there has to be a local correspondence between the two flows, then (3.13) defines, for a given distribution of Pr_t, a corresponding distribution of $\overline{Pr}_t$ for the transformed flow. Hence the relations (3.8) imply a well-defined transformation of the turbulent Prandtl number distribution. If we follow the reasoning of Coles, the transformed distribution must correspond to a physical reality. On the other hand, this may be too much to ask from a transformation which is of a purely mathematical nature.

Observe that at the wall, where the molecular Prandtl numbers have been assumed to be identical in the two flows, the relation (3.13) can be shown to be automatically satisfied if the expression in brackets in the second member of (2.25) vanishes, as it must do if η/σ must be constant as assumed.

In concluding the constant-pressure case, we notice that the transformation is completely defined without any need of prescribing the physical properties of the low-speed fluid, except at the wall. In other words, we obtain the same transformation if the low-speed fluid is chosen to be a liquid of constant density or a gas with the density determined only by the temperature, by virtue of the low-speed assumption.

4. Variable-Pressure Case

As soon as a pressure gradient is introduced, the choice of the properties of the low-speed fluid becomes essential in determining the transformation, as is immediately apparent from (2.25) where neither of the terms of the second member vanishes, as it does in the constant pressure case.

We shall, therefore, particularize our transformation for the two cases just mentioned in view of their practical physical significance. If both fluids are perfect polytropic gases, (2.25) can be obtained in a particularly simple form as follows. For perfect polytropic gases, one has, recalling (2.12),

$$\frac{\bar{\rho}_e}{\bar{\rho}} = \frac{\bar{h}}{\bar{h}_e} = \frac{h^0}{h_e^0} \qquad \frac{\rho_e}{\rho} = \frac{h}{h_e} \tag{4.1}$$

where it has been assumed not only that the specific heats are constant, but also that the additive constant of the enthalpy has been chosen so as to make the enthalpy proportional to the absolute temperature. As a result of (4.1), one can calculate the integral appearing in (2.25)

$$\int_0^\delta\left(1 - \frac{\bar{\rho}_e}{\bar{\rho}}\frac{\rho}{\rho_e}\right)dy = \frac{1}{\rho_e h_e^0}\int_0^\delta\left[\rho_e\left(h_e + \frac{u_e^2}{2}\right) - \rho\left(h + \frac{u^2}{2}\right)\right]dy$$
$$= \frac{1}{\rho_e h_e^0}\int_0^\delta\left(\rho_e\frac{u_e^2}{2} - \rho\frac{u^2}{2}\right)dy$$
$$= \frac{u_e^2}{2h_e^0}(\delta^* + \theta) \tag{4.2}$$

as is immediately clear from the definitions (2.3).

As a consequence of (4.2), one can write

$$\delta^* + \theta - \int_0^\delta \left(1 - \frac{\bar{\rho}_e}{\bar{\rho}}\frac{\rho}{\rho_e}\right) dy = \left(1 - \frac{u_e^2}{2h_e^0}\right)(\delta^* + \theta) = \frac{h_e}{h_e^0}(\delta^* + \theta)$$

and

$$\frac{1}{u_e}\frac{du_e}{dx}\int_0^\delta \left(1 - \frac{\bar{\rho}_e}{\bar{\rho}}\frac{\rho}{\rho_e}\right) dy = \frac{u_e}{2h_e^0}\frac{du_e}{dx}(\delta^* + \theta) = -\frac{1}{2h_e^0}\frac{dh_e}{dx}(\delta^* + \theta)$$

Hence (2.18) can be written in the concise form

$$\xi = \frac{\sigma^2}{\eta}\frac{\tau_w}{\bar{\tau}_w}\left[1 + \frac{2}{c_f}\theta\frac{d}{dx}(\ln\sigma) - \frac{h_e}{h_e^0}(\delta^* + \theta)\frac{d}{dx}\left\{\ln\left[\frac{\eta}{\sigma}\left(\frac{h_e^0}{h_e}\right)^{1/2}\right]\right\}\right] \quad (4.3)$$

Since (2.23) is unchanged, (2.25) can be obtained in the form

$$(\delta^* + \theta)\frac{d}{dx}\left\{\ln\left[\frac{\eta}{\sigma}\left(\frac{h_e^0}{h_e}\right)^{1/2}\right]\right\} = \frac{h_e^0}{h_e}\left(\theta - \frac{h_e^0 - h_w}{h_r - h_w}\frac{c_f}{2c_h}\varphi\right)\frac{d}{dx}(\ln\sigma) \quad (4.4)$$

We notice immediately that (4.3) becomes particularly simple if

$$\frac{\eta}{\sigma}\left(\frac{h_e^0}{h_e}\right)^{1/2} = \frac{u}{\bar{u}}\left(\frac{h_e^0}{h_e}\right)^{1/2} = \text{const} \quad (4.5)$$

This is actually the form of the velocity transformation which has been used by most authors, as it results directly from the extension of the Stewartson-Illingworth transformation to turbulent flows. The reason for the widespread use of (4.5) is seen to reside in the simplification it induces in (4.3). However, one sees from (4.4) that (4.5) is consistent only with the constant-pressure case for which, as already seen, the expression in parentheses of the second member of (4.4) vanishes, or with the case σ = const, which would induce a further simplification in (4.3) but has to be discarded on grounds of experimental evidence, as Coles has clearly pointed out, and as will be seen in Sec. 5.

Hence Eq. (4.4) shows that (4.5) cannot be true in the variable-pressure case, and provides the right expression for the calculation, by integration, of the proper values of $(\eta/\sigma)(h_e^0/h_e)^{1/2}$ as a function of x, when both fluids are perfect polytropic gases and $\sigma(x)$ is known. Next, the values of $\xi(x)$ can be obtained by means of (2.23) or (4.3), and the transformation is completed. This transformation will be called the "low-velocity transformation," or briefly the "l transformation" to characterize its main feature, which is suppressing the disturbing effect of the high velocities without changing the state behavior of the fluid.

If the l transformation must hold locally throughout the boundary layer, one needs to evaluate the various terms of (2.27). Proceeding in the same fashion as for (4.2), we obtain

$$\int_y^\delta \left(1 - \frac{\bar{\rho}_e}{\bar{\rho}}\frac{\rho}{\rho_e}\right) dy = \frac{u_e^2}{2h_e^0}\int_y^\delta \left(1 - \frac{\rho u^2}{\rho_e u_e^2}\right) dy$$

and

$$\int_y^\delta \frac{\rho}{\rho_e}\left(\frac{u^2}{u_e^2} - \frac{\bar{\rho}_e}{\bar{\rho}}\right) dy = -\frac{h_e}{2h_e^0}\int_y^\delta \left(1 - \frac{\rho u^2}{\rho_e u_e^2}\right) dy$$

Hence, after substitution of (4.4), (2.27) becomes

$$\bar{\tau} = \frac{\sigma^2}{\xi\eta}\left\{\tau + \frac{1}{r^\epsilon}\frac{d}{dx}(\ln\sigma)\left[\int_y^\delta \psi\frac{\partial u}{\partial y}dy - \frac{\rho_e u_e^2}{\theta + \delta^*}\left(\theta - \frac{h_e^0 - h_w}{h_r - h_w}\frac{c_f}{2c_h}\varphi\right)\int_y^\delta\left(1 - \frac{z^{n\sigma}}{\rho_e u_e^2}\right)dy\right]\right\} \quad (4.6)$$

Since Eq. (2.29) is the same as for the constant-pressure flow, as are Eqs. (3.10), proceeding in the same way we end up with a relation similar to (3.13), where, however, the expression in parentheses in the first member is replaced by the expression in brackets in the second member of (4.6). Again, we find that the Prandtl number must be transformed according to a well-determined law. However, contrary to what happens in the constant-pressure case, there is no particularly simple law for the transformed Prandtl number when the original Prandtl number is unity. (This is perhaps the strongest argument for doubting the local validity of the transformation.)

The author must confess that he has a particular inclination for the l transformation, as compared to the other transformation to be discussed next.# With the exception of the case $q_w = 0$, $Pr_t = 1$, for which $\bar{h}$ is constant across the boundary layer, and for which we obtain a constant-density low-speed flow when heat exchange is present, the l transformation makes the proper distinction between the effect of the high velocity and that of the heat flux in determining the density distribution of the fluid. The author has the impression that it is physically more realistic to establish a correspondence between two flows of the same substance, so that the effects of the heat exchange in determining the transverse spacing of the fluid elements are preserved, rather than look for a correspondence between two different types of fluids, one of which does not contain these effects. But, of course, this intuitive justification does not represent a proof. Moreover, the l transformation presents a great inconvenience; very little empirical information exists on the low-velocity, turbulent boundary-layer flow of a gas with heat exchange. Hence, even if the approach appears to be more attractive, it cannot be used before more information becomes available about velocity and temperature distributions and similarity correlations in this kind of flow.

On the other hand, a substantial amount of experimental research has been developed for adiabatic low-velocity flows where the fluid is either a liquid or a (constant-density) gas. If the fluid is a liquid, the only effect of the heat exchange is felt through the variation of viscosity with temperature. For an ideal liquid with viscosity independent of the temperature,** this effect disappears, and all the information for adiabatic constant-density flows becomes applicable to constant-density flows with heat transfer and, hence, through the proper transformation, to high-velocity flows of gases with heat exchange.

Therefore, from a practical point of view, it is particularly interesting to define a transformation where the low-velocity fluid has constant density and constant viscosity, even if $\bar{h}$ is variable. We shall call this transformation the "incompressible transformation" or briefly the "i transformation" to indicate the main feature of constant density of the low-speed fluid.

The relations defining the i transformation can be determined from the general relations of Sec. 2 by taking $\bar{\rho}$ =

I observe that, in the laminar case with heat exchange, it is impossible to find a transformation from a high-velocity gas to a low-velocity liquid, even for $Pr = 1$, $\mu \sim T$. Under these assumptions, on the contrary, the high-speed flow of a perfect gas is transformed into a low-speed flow of a perfect gas.

** The effect of the viscosity variation in a constant-density fluid can, on the other hand, be taken into account via a particular choice of $\sigma(x)$, as it is actually taken into account in gaseous flows following the procedure of Coles to be discussed in Sec. 5.

$\bar{\rho}_e$. The integral appearing in these relations reduces to the expression

$$\varphi^* = \int_0^\delta \left(1 - \frac{\rho}{\rho_e}\right) dy = \int_0^\delta \left(1 - \frac{h_e}{h}\right) dy \qquad (4.7)$$

and defines a new thickness which might be called the "density thickness." The last expression is obtained under the assumption that the high-speed fluid is a perfect, polytropic gas. For the general type of flow, the evaluation of this thickness cannot be made explicitly because, in order to obtain the enthalpy field, one must, even if the velocity field is known, carry out the solution of the energy equation.

The difficulty arising from the lack of a closed-form solution of the energy equation can be overcome using an approximate procedure introduced by Cohen.[9] The idea is to correct the linear relation between the stagnation enthalpy and the velocity, valid for $Pr_t = 1$ in the constant-pressure case, in such a way as to take care approximately of the effects of pressure gradients and Prandtl number. The simplest kind of correction to a linear relation is the addition of a quadratic term. Hence with Cohen we take

$$\frac{h^0}{h_e^0} = \frac{h_w}{h_e^0} + \left(1 - \frac{h_w}{h_e^0} - c\right)\frac{u}{u_e} + c\left(\frac{u}{u_e}\right)^2 \qquad (4.8)$$

where $c(x)$ is a correction coefficient to be determined, vanishing for $Pr_t = 1$ in the constant-pressure case.

Making use of (4.8) in (4.7) and assuming, as previously, $\rho/\rho_e = h_e/h$, one obtains, after manipulations,

$$\varphi^* = \left(1 - \frac{h_e}{h_w}\right)\delta^* + \left[(1-c)\frac{h_e^0}{h_w} - \frac{h_e}{h_w}\right]\theta \qquad (4.9)$$

Moreover, introducing (4.8) in (2.16) and (2.21), one obtains

$$\frac{q_w}{\tau_w} = \frac{1}{Pr_w}\frac{h_e^0(1-c) - h_w}{u_e}$$

so that the Reynolds analogy factor is given by

$$s = \frac{c_f}{2c_h} = Pr_w \frac{h_r - h_w}{h_e^0(1-c) - h_w} \qquad (4.10)$$

and, therefore, can be calculated if $c(x)$ and the recovery factor are known.

The final expression obtained from (2.25) is

$$\frac{d}{dx}\left(\ln\frac{\eta}{\sigma}\right) = \frac{\frac{d}{dx}(\ln\sigma)\left[\theta - Pr_w\frac{h_e^0 - h_w}{h_e^0(1-c) - h_w}\varphi\right] - \frac{d}{dx}(\ln u_e)\left\{\left(1 - \frac{h_e}{h_w}\right)\delta^* - \left[\frac{h_e}{h_w} - (1-c)\frac{h_e^0}{h_w}\right]\theta\right\}}{\frac{h_e}{h_w}\delta^* + \left[1 + \frac{h_e}{h_w} - (1-c)\frac{h_e^0}{h_w}\right]\theta}$$

The expressions (2.18) and (2.23) for $\xi(x)$ remain unchanged with the proper value of the common factor in front of the brackets, that is,

$$\frac{\sigma^2}{\eta}\frac{\tau_w}{\bar{\tau}_w} = \sigma\frac{\bar{h}_e}{h_e^0}\frac{q_w}{\bar{q}_w} = \eta\sigma\frac{\rho_w\mu_w}{\bar{\rho}\bar{\mu}}$$

A somewhat more sophisticated expression than (4.8) can be obtained by adding a cubic term

$$h^0/h_e^0 = h_w/h_e^0 + bu/u_e + c(u/u_e)^2 + d(u/u_e)^3 \qquad (4.11)$$

The functions $b(x)$, $c(x)$, and $d(x)$ must satisfy the relation

$$b + c + d = 1 - h_w/h_e^0 \qquad (4.12)$$

The extra coefficient introduced allows the fulfillment of an additional condition. For instance, in view of the particular importance of the quantities at the wall, one can choose to satisfy the energy equation (2.28) at $y = 0$, or Eq. (3.3) at $u = 0$, that is,

$$\tau_w\left(\frac{\partial^2 h}{\partial u^2}\right)_w + (1 - Pr_w)\left(\frac{\partial\tau}{\partial u}\frac{\partial h}{\partial u}\right)_w + Pr_w\frac{u_w}{\tau_w}\frac{dp}{dx}\left(\frac{\partial h}{\partial u}\right)_w = 0 \qquad (4.13)$$

Here we have used, at the wall, the laminar relation $(\partial y/\partial u)_w = \mu_w/\tau_w$. From the momentum equation at the wall, $(\partial\tau/\partial y)_w = dp/dx$, one obtains

$$(\partial\tau/\partial u)_w = (\mu_w/\tau_w)(dp/dx)$$

This relation and Eq. (4.11) introduced in (4.13) provide the following relation between $c(x)$ and $b(x)$:

$$c + kb = (1 - Pr_w)\left[1 - \frac{h_e}{h_e^0}\right] \qquad (4.14)$$

with

$$k = \frac{\mu_w u_e}{2\tau_w^2}\frac{dp}{dx} = \frac{2\mu_w}{\rho_e c_f^2}\frac{d}{dx}\left(\frac{1}{u_e}\right) \qquad (4.15)$$

Equations (4.12) and (4.14) allow us to express c and b as functions of $d(x)$. The consequent calculation of φ^* using (4.11) produces the result

$$\varphi^* = \left(1 - \frac{h_e}{h_w}\right)\delta^* + \frac{1}{1-k}\left[Pr_w\frac{h_e^0 - h_e}{h_w} - k\left(1 - \frac{h_e}{h_w} + \frac{h_e^0}{h_w}d\right)\right]\theta - \frac{h_e^0}{h_w}d\delta^{**} \qquad (4.16)$$

Here

$$\delta^{**} = \int_0^\delta \frac{\rho u}{\rho_e u_e}\left(1 - \frac{u^2}{u_e^2}\right)dy \qquad (4.17)$$

is the "kinetic energy" thickness used by Truckenbrodt.[14]

The equation corresponding to (4.10) is

$$s = \frac{c_f}{2c_h} = Pr_w\frac{h_r - h_w}{h_e^0 b} = \frac{Pr_w(h_r - h_w)(1-k)}{(Pr_w - d)h_e^0 + (1 - Pr_w)h_e - h_w} \qquad (4.18)$$

Equations (4.16) and (4.18), when introduced in (2.15), provide the proper relation for the determination of η/σ, after which ξ can be obtained.

One sees that the introduction of the cubic term in the stagnation energy relation results in more complicated, though not unmanageable, expressions. In particular, it introduces the kinetic energy thickness δ^{**}. Observe that this is not too much of an inconvenience, particularly if in solving the integral boundary-layer equation one follows Truckenbrodt procedure of taking into consideration the equation for the conservation of mechanical energy which, in integral form, contains the same δ^{**}. However, one must notice that the use of polynomials of higher degree in the approximate solution of the integral boundary-layer equation does not necessarily improve the accuracy, a well-known fact for the Pohlhausen method.

Before closing this section, the author would like to point out the implications of the existence of two different transformations, both resulting in low-speed flows of a fluid with a well-defined physical significance. One can, with Coles, postulate that, if a transformation satisfies both the requirement of correctly transforming the inertia and pressure terms and that of predicting the right behavior for the friction coefficient, then the transformation correctly transforms the local values of the various quantities and establishes a real correspondence between two flows. In this case, it is not surprising to find the existence of two different transformations. In fact, if a high-speed flow with heat exchange can be made to correspond precisely to an incompressible flow through an i transformation, so can a low-speed gaseous flow with heat transfer. If the resulting incompressible flow is the same in both cases, clearly an l transformation will exist which establishes a direct correspondence between the two gaseous flows. It is evident that in this case there will be every advantage in using the i transformation, and not the l transformation. However, if, contrary to Coles' postulate, the transformations are only mathematical artifices proper to operate on the integral boundary-layer equations, but not valid locally, then one of the transformations may work better than the other from a local standpoint. This is the interpretation toward which the author personally inclines with a preference, as previousy discussed, for the l transformation.

5. Brief Considerations on the Choice of $\sigma(x)$ and Other Matters

Coles has obtained for σ a simple expression that consistently transforms, with very little scatter, the skin-friction coefficient (obtained by several authors with the floating-element technique for high-speed, constant-pressure flows without heat exchange) into the low-speed friction coefficient. Here, without following Coles' derivation, the author will take a different path to show the essence of his results. Let us start from the observation that

$$Re_\psi = \frac{1}{r^\epsilon}\int_0^\psi \frac{d\psi'}{\mu} = \int_0^y \frac{\rho u dy'}{\mu} \tag{5.1}$$

is an average Reynolds number for any given layer of the flow. For each layer one can also define a mean viscosity μ_m:

$$\frac{1}{\mu_m} = \frac{1}{\psi}\int_0^\psi \frac{d\psi'}{\mu} \tag{5.2}$$

The assumption is now made, with Coles, that there exists a certain turbulent substructure, corresponding to a well-defined value

$$Re_{\psi_s} = \frac{1}{r^\epsilon}\int_0^{\psi_s} \frac{d\psi}{\mu} = \frac{\psi_s}{r^\epsilon \mu_{ms}} \tag{5.3}$$

of the just-defined Reynolds number, and that outside of this substructure the viscosity that counts for the behavior of turbulence is not the mean value μ_m defined by (5.2) for the whole layer, but the mean value μ_{ms} for the substructure alone. Then the relevant Reynolds number is not (5.1), that is, $\psi/\mu_m r^\epsilon$, but $\psi/\mu_{ms} r^\epsilon$, and this is the quantity which must remain invariant against a transformation establishing a correspondence between two flows. If the i transformation is used, this means that

$$\psi/r^\epsilon \mu_{ms} = \bar{\psi}/\bar{r}^\epsilon \bar{\mu} \tag{5.4}$$

an equation which, when compared with the first (2.9), provides the relation

$$\sigma = \bar{\mu}/\mu_{ms} \tag{5.5}$$

which is Coles' relation, except for the definition of μ_{ms}.

If the l transformation is used, $\bar{\mu}$ is not a constant and must be replaced by the mean value $\bar{\mu}_{ms}$ given by the relation (5.3), and applied to the transformed flow.

The only point in which these derivations differ from those of Coles is precisely in the expression used to define the mean viscosity. Coles' definition is as follows. If y_s is the thickness of the substructure, then the following substructure Reynolds number

$$Re_s = \frac{\rho_s u_s y_s}{\mu_s} \tag{5.6}$$

must be a constant, when u_s is the actual velocity at the edge of the sublayer, and ρ_s, μ_s are properly defined mean values. Coles has chosen

$$\rho_s = \frac{1}{y_s}\int_0^{y_s} \rho dy \tag{5.7}$$

For a perfect gas, ρ_s corresponds to a temperature T_s and a viscosity μ_s given by

$$\frac{1}{T_s} = \frac{1}{y_s}\int_0^{y_s}\frac{dy}{T} \qquad \mu_s = \mu(T_s) \tag{5.8}$$

As Coles himself has recognized, this choice is quite arbitrary.

On the other hand, after the i transformation, one has

$$Re_s = \bar{\rho}\bar{u}_s\bar{y}_s/\bar{\mu} \tag{5.9}$$

If the "law of the wall" is assumed to be valid, one can apply the corresponding universal relation,

$$\bar{u}/\bar{u}_\tau = f(\bar{\rho}\bar{u}_\tau\bar{y}/\bar{\mu}) = f(z) \tag{5.10}$$

with $\bar{u}_\tau = (\bar{\tau}_w/\bar{\rho})^{1/2}$. Hence, the value of

$$Re_s = z_s f(z_s)$$

defines the value of z_s and $f(z_s)$. From comparison with the experimental results, Coles finds an excellent fit for

$$Re_s = 8500 \qquad z_s = 430 \qquad f(z_s) = \bar{u}_s/\bar{u}_\tau = 19.8 \tag{5.11}$$

At first it would seem that there is a considerable difference between (5.3) and (5.6). A closer analysis shows that this is not true. Indeed, introducing a mean density ρ_{ms}, viscosity μ_{ms}, and velocity u_{ms}, (5.3) can be written

$$Re_{\psi_s} = (\rho_{ms}u_{ms}y_s)/\mu_{ms} \tag{5.12}$$

In nonuniform flows, the mean quantities can be defined in many different ways. It is, however, a general property[15] that, for increasingly uniform flows, the mean values (of a given quantity) defined in different ways tend to come closer together until, for a sufficient degree of uniformity, it matters very little how the mean value is calculated.

Now, the turbulent flow in the substructure is quite uniform because of the large value of Re_s. As an example of the practical coincidence of mean values calculated for the incompressible substructure in different ways, take the two following mean velocities

$$\bar{u}_{ms}' = \frac{\int_0^{\bar{y}_s} \bar{u}^2 d\bar{y}}{\int_0^{\bar{y}_s} \bar{u} d\bar{y}} \qquad (\bar{u}_{ms}'')^2 = \frac{\int_0^{\bar{y}_s} \bar{u}^3 d\bar{y}}{\int_0^{\bar{y}_s} \bar{u} d\bar{y}} \tag{5.13}$$

the first calculated from the momentum and the mass flux, the second from the kinetic energy and mass flux. Introducing the law of the wall in the particular form

$$\bar{u}/\bar{u}_\tau = f(z) = (1/\varkappa)\ln z + c \tag{5.14}$$

(with $\varkappa = 0.41$, $c = 5.00$) and neglecting, for the present purpose, the deviation from (5.14) near the wall, one calculate from (5.13)

$$\varkappa\frac{\bar{u}_{ms}'}{\bar{u}_\tau} = \varkappa\frac{\bar{u}_s}{\bar{u}_\tau} - 1 + \frac{1}{\varkappa(\bar{u}_s/\bar{u}_\tau) - 1}$$

$$\left(\varkappa\frac{\bar{u}_{ms}''}{\bar{u}_\tau}\right)^2 = \left(\varkappa\frac{\bar{u}_s}{\bar{u}_\tau} - 1\right)^2 + 3 - \frac{2}{\varkappa(\bar{u}_s/\bar{u}_\tau) - 1}$$

and we find, for the value (5.11) of $\bar{u}_s/\bar{u}_r$,

$$\bar{u}_{ms}'/\bar{u}_s = 0.895 \qquad \bar{u}_{ms}''/\bar{u}_s = 0.900 \tag{5.15}$$

which proves our point. What is true for incompressible flows is also true for compressible flows, where it does not matter much how the mean temperature T_{ms} is calculated (for instance, from the mass flux using the equation of state or from the enthalpy flux).[15] Within the same accuracy it is also true that the relation between mean thermodynamic variables can be taken to be the same as between local values, so that it makes very little difference if μ_{ms}, instead of being calculated from the local μ values, is just taken to be $\mu(T_{ms})$. As a result, one sees that we can safely identify ρ_{ms} and μ_{ms} with Coles' ρ_s and μ_s given by (5.7) and (5.8), and write (5.12) as follows:

$$Re_{\psi_s} = \frac{\rho_s u_{ms} y}{\mu_s} \tag{5.16}$$

Comparing with (5.6), one sees that

$$Re_{\psi_s}/Re_s = u_{ms}/u_s = \bar{u}_{ms}/\bar{u}_s$$

has, according to (5.15), an approximate value of about 0.900, so that the condition (5.11) corresponds to

$$Re_{\psi_s} = 7650$$

with very good accuracy.

This statement obviously can be reversed, and the condition of constant Re_{ψ_s} can be replaced by the conditions (5.11) used by Coles.††

The actual determination of T_s or h_s can be made by rewriting, as Coles did, (5.8) in the form

$$h_s = \frac{1}{z_s}\int_0^{z_s} h dz$$

Coles used this equation in the case when h^0 is a linear function of u.‡‡ Clearly, it can be used without additional complication in conjunction with the i transformation using the relations (4.8) and (5.10) with results similar in form to those of Coles [the expression for h_s contains in this case $c(x)$] or making use of (4.11) and (5.10) with results somewhat more complicated than those of Coles [the resulting expression for h_s would depend in this case on $d(x)$].

If the l transformation were used, the same procedure could, in principle, be applied, but would stumble on the difficulty that the form of the law of the wall is not known for low-velocity flows of gases with heat exchanges. Again one sees that the l transformation cannot be used until more empirical knowledge is accumulated on this type of flow.

†† We observe that Coles himself has suggested the alternate use of $\int_0^{z_s} f dz = Re_{\psi_s}$ for the substructure Reynolds number, pointing out the computational complications that this definition would introduce. Our discussion shows that, even if not rigorously, from a practical point of view the two definitions are equivalent.

‡‡ As previously noticed, this is the only point where the local validity of the transformation is assumed in expressing the relation between h and z.

The practical way of solving the turbulent boundary-layer flow around a body moving at high speeds consists, of course, of first transforming to the corresponding low-velocity flow which, at the present stage of accumulated experimental knowledge, should be incompressible, and secondly, finding the solution to this low-velocity flow. The discussion of this question is out of the scope of the present paper. The author only will make the remark that, in case the problem is attacked via the Truckenbrodt procedure of making use of the integral equation for the conservation of mechanical energy, it should be clear that this equation also obeys to the same transformation as the others if Coles' idea of a local correspondence is verified, so that the procedure would maintain its physical consistence. However, it is the author's feeling that, although the formulation of Truckenbrodt's procedure is perfectly legitimate, the use of a dissipation coefficient which cannot be determined on firm grounds[16] represents a clear weakness of this procedure. Other concepts might perhaps be used with advantage, some of them already contemplated by other authors, some still waiting in the nebulous future to be discovered.

References

[1] van Driest, E. R., "Turbulent boundary layer in compressible fluids," J. Aeronaut. Sci. **18**, 145–160 (1951).

[2] Donaldson, C. duP., "On the form of the turbulent skin-friction law and its extension to compressible flows," NACA TN 2692 (1952).

[3] Rotta, J. C., "Turbulent boundary layers with heat transfer in compressible flow," AGARD Rept. 281 (April 1960).

[4] Ferrari, C., *Corso Sulla Teoria della Turbolenza* (Edizioni Cremonese, Roma, 1957), Sec. C; also transl. as NASA Re 2-8-59W (March 1959).

[5] Townsend, A. A., *The Structure of Turbulent Shear Flow* (Cambridge University Press, New York, 1956).

[6] Kalikhman, L. E., "Heat transmission in the boundary layer," NACA TM 1229 (1949).

[7] Mager, A., "Transformation of the compressible turbulent boundary layer," J. Aerospace Sci. **25**, 305–311 (1958).

[8] Englert, G. W., "Estimation of compressible boundary-layer growth over insulated surfaces with pressure gradient," NACA TN 4022 (1957).

[9] Cohen, N. B., "A method of computing turbulent heat transfer in the presence of a streamwise pressure gradient for bodies in high-speed flow," NASA Memo. 1-2-59L (1959).

[10] Coles, D. E., "The turbulent boundary layer in a compressible fluid," Rand Corp. Rept. R-403-PR (1962).

[11] Crocco, L., "Su di un valor massimo del coefficiente di trasmissione del calore da una lamina piana a un fluido scorrente," L'Aerotecnica **12**, 181–197 (1932).

[12] Crocco, L., "Lo strato limite laminare nei gas," Monografie Scientifiche d'Aeronautica, no. 6 (1947).

[13] van Driest, E. R., "Convective heat transfer in gases," High Speed Aerodynamics and Jet Propulsion **5**, Sec. F, 339–427 (1959).

[14] Truckenbrodt, E., "Ein Quadraturverfahren zur Berechnung der Laminaren und Turbulenten Reibungsschicht bei Ebener und Rotationssymmetrischer Stromung," Ing. Arch. **20**, 211–228 (1952).

[15] Crocco, L., "Validity of the uniformity assumption," High Speed Aerodynamics and Jet Propulsion **3**, Sec. B, 293–348 (1958).

[16] Rotta, J., "Schubspannungsverteilung und Energiedissipation bei Turbulenten Grenzschichten," Ing. Arch. **20**, 195–207 (1952).